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OPTIMIZATION OF ELASTIC BARS IN TORSION

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Abstract—The paper considers the problem of optimization of mechanical systems described by partial differential equations. The shape of the region of integration of these equations is not specified beforehand but is to determined from the condition that a certain integral functional attains an extremal value. The mathematical optimization problem is reduced to a variational one having no differential constraints and the necessary optimality conditions are derived. The latter are used for seeking the cross-sectional shape of elastic bars of maximum torsional rigidity. Exact and approximate analytical solutions are given and the effectiveness of the optimal solutions is estimated.

INTRODUCTION

The paper deals with the problem of finding the cross-sectional shape of elastic bars that has the maximum torsional rigidity. As well as having a direct practical application such problems are of great interest from a purely mathematical angle, viz. from the point of view of developing effective analytical methods for optimizing mechanical systems that are described by (elliptical) partial differential equations.

The problem of optimizing the cross-sectional shape of an elastic bar under torsion was first considered by Polya (the results of his investigations are contained, for example, in [1]). Employing the symmetrization theorem he showed that among all the bars of a simply-connected and convex cross-section the circular bar has the maximum torsional rigidity.

Klosowicz and Lurie[2] considered the optimal design problem of bars made of two elastic materials with different shear moduli. It should be mentioned that these authors gave solutions for the optimal distribution of the materials along the cross-section having the maximum or the minimum torsional rigidity and revealed qualitatively certain characteristic features of the optimal solutions.

In the present paper we consider the optimal design problem of anisotropic elastic bars and of homogeneous elastic bars with a multiply-connected (hollow) cross-section. Initially we study a more general mathematical optimization problem for a class of (elliptical) partial differential equations and derive the necessary optimality conditions. The latter are then used for solving the optimal design problem of an anisotropic elastic bar under torsion. It is shown that the bar with maximum torsional rigidity will have an elliptical cross-section.

Next we solve problems of optimization of thin, homogeneous isotropic bars with a doubly-connected cross-section using a perturbation technique (method of small parameter). Characteristic features of the optimal shapes are revealed and the effectiveness of optimal solutions estimated. The perturbation technique used here allows us to obtain analytical solutions (with any desired degree of accuracy) and can be used for the optimal design of homogeneous (heterogeneous), isotropic (anisotropic) bars of a multiply-connected cross-section.

1. VARIATIONAL PROBLEM FOR A REGION WITH UNKNOWN BOUNDARY

Let the function ϕ satisfy, in a simply-connected region D, the partial differential equation

$$(a\phi_x - c\phi_y)_x + (b\phi_y - c\phi_x)_y + m = 0, \qquad (x, y) \in D \tag{1.1}$$

and the boundary condition

$$\phi = 0, \qquad (x, y) \in \Gamma \tag{1.2}$$

on the contour Γ bounding the region D. The coefficients $a = a(x, y)$, $b = b(x, y)$, $c = c(x, y)$ are

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given, continuously differentiable functions satisfying the requirement

$$a > 0, \quad ab - c^2 > 0, \tag{1.3}$$

and m is a given constant. Subscripts denote differentiation with respect to the corresponding spatial coordinate.

The area of the region D is denoted by S

$$\iint_D \mathrm{d}x\, \mathrm{d}y = S, \tag{1.4}$$

and is supposed to be given.

The problem under consideration consists in finding the shape of the contour Γ that satisfies the isoperimetric condition (1.4) and for which the functional $K(\Gamma)$ achieves a maximum

$$K(\Gamma) = \iint_D \phi\, \mathrm{d}x\, \mathrm{d}y \to \max. \tag{1.5}$$

The optimization problem formulated above is one with a differential constraint (1.1). Let us reduce it to a variational problem with no such constraints. (Such problems have been dealt with earlier, see, for example, [3]). In this connection, consider the following. As is well known, if the boundary contour Γ is given and the condition (1.3) is satisfied, the function ϕ, being the solution of the boundary value problem (1.1)–(1.2), reduces the integral function to a minimum

$$J = \iint_D (a\phi_x^2 - 2c\phi_x\phi_y + b\phi_y^2 - 2m\phi)\, \mathrm{d}x\, \mathrm{d}y. \tag{1.6}$$

The converse is also true, i.e. the function ϕ which reduces the functional J to a minimum is a solution of the boundary value problem (1.1)–(1.2).

We now show that, for a function ϕ which reduces the functional (1.6) to a minimum subject to the condition (1.2), the following equality holds good

$$J = -mK. \tag{1.7}$$

Using conditions (1.1), (1.2) and (1.5), we can write the following

$$-mK = -\iint_D \phi m\, \mathrm{d}x\, \mathrm{d}y = -\iint_D \phi[(a\phi_x - c\phi_y)_x + (b\phi_y - c\phi_x)_y]\, \mathrm{d}x\, \mathrm{d}y$$

$$= -\iint_D [\phi_x(a\phi_x - c\phi_y) + \phi_y(b\phi_y - c\phi_x)]\, \mathrm{d}x\, \mathrm{d}y,$$

whence follows the required relation

$$J = \left[\iint_D (a\phi_x^2 - 2c\phi_x\phi_y + b\phi_y^2)\, \mathrm{d}x\, \mathrm{d}y - m\iint_D \phi\, \mathrm{d}x\, \mathrm{d}y\right]$$

$$- m\iint_D \phi\, \mathrm{d}x\, \mathrm{d}y = -m\iint_D \phi\, \mathrm{d}x\, \mathrm{d}y = -mK. \tag{1.8}$$

The above formula (1.8) is a direct generalization of the corresponding equality, well-known in the theory of elastic bars under torsion [6]. Using the variational principle and the relation (1.8), we have

$$K = -\frac{1}{m}\min_{\phi}\iint_{D} (a\phi_x^2 - 2c\phi_x\phi_y + b\phi_y^2 - 2m\phi)\,dx\,dy. \tag{1.9}$$

This expression allows us to reduce the initial optimization problem (1.1)-(1.5) to a variational one with no differential constraints

$$\begin{aligned} K_* &= \max_{\Gamma} K(\Gamma) \\ &= \max_{\Gamma}\min_{\phi}\frac{1}{m}\iint_{D} (a\phi_x^2 - 2c\phi_x\phi_y + b\phi_y^2 - 2m\phi)\,dx\,dy \end{aligned} \tag{1.10}$$

It may be noted that the minimum with respect to ϕ in (1.10) is sought among a class of continuously differentiable functions satisfying the boundary condition (1.2), while the maximum with respect to Γ is sought subject to the isoperimetric condition (1.4).

Let us derive the necessary optimality conditions determining the boundary of the region D. In this connection we write the expression for the first variation of the functional J

$$\begin{aligned} \delta J = &-2\iint_{D} \delta\phi[(a\phi_x - c\phi_y)_x + (b\phi_y - c\phi_x)_y + m]\,dx\,dy \\ &+ 2\int_{\Gamma} \delta\phi[(a\phi_x - c\phi_y)\,dy - (b\phi_y - c\phi_x)\,dx] \\ &+ \int_{\Gamma} \delta f(a\phi_x^2 + b\phi_y^2 - 2c\phi_x\phi_y - 2m\phi)\,ds, \end{aligned} \tag{1.11}$$

where δf denotes the normal displacement of points on the contour Γ resulting from a variation of the region D. As the function ϕ is given on the contour Γ (it satisfies the boundary condition (1.2)), $\delta\phi = 0$, and consequently the second integral on the right hand side of (1.11) drops out. From the requirement that $\delta J = 0$ (i.e., the necessary condition for extremum), the arbitrariness of $\delta\phi$ in the region D and the boundary condition (1.2), we arrive at the eqn (1.1) and the equality

$$\int_{\Gamma} \delta f(a\phi_x^2 - 2c\phi_x\phi_y + b\phi_y^2)\,ds = 0. \tag{1.12}$$

Furthermore, by taking into account the fact that δf satisfies the equation

$$\int_{\Gamma} \delta f\,ds = 0 \tag{1.13}$$

which follows from the isoperimetric condition (1.4), and by using (1.12), we get

$$a\phi_x^2 - 2c\phi_x\phi_y + b\phi_y^2 = \lambda^2. \tag{1.14}$$

The equality (1.14) is the required optimality condition and serves to define the boundary Γ and implies that the complementary strain energy density on the boundary is constant—in conformity with similar optimality condition derived by Prager [4] and Masur [5]. λ^2 is an arbitrary constant to be determined from the isoperimetric condition (1.4). Thus the shape of the optimal contour Γ and the corresponding function $\phi(x, y)$ may be found from the relations (1.1), (1.2), (1.4) and (1.14).

2. OPTIMAL SHAPE OF AN ANISOTROPIC BAR UNDER TORSION

Consider the problem of an elastic bar under torsion. Let the bar lie along the axis z in a rectangular cartesian coordinate system xyz and be subjected to torsional moments M at its ends.

Let D denote the cross-section of the bar perpendicular to z-axis and Γ the boundary of the region D. We introduce a stress function $\phi = \phi(x, y)$ and express through it the non-vanishing components T_{yz} and T_{xz} of the stress tensor

$$T_{xz} = \theta\phi_y, \qquad T_{yz} = -\theta\phi_x, \tag{2.1}$$

where θ is the angle of twist per unit length of the bar.

The stress function, as is well known, is determined by solving the partial differential equation

$$a\phi_{xx} - 2c\phi_{xy} + b\phi_{yy} = -2 \tag{2.2}$$

subject to the boundary conditions (1.2). Here, a, b and c denote the elastic constants of the anisotropic material of the bar.

The torsional rigidity K is calculated from ϕ

$$K = 2\iint_D \phi \, dx \, dy. \tag{2.3}$$

It may be noted, that the twisting moment M, torsional rigidity K and the angle of twist θ are related through

$$M = K\theta.$$

The optimization problem considered here consists in seeking the contour Γ bounding the region D that satisfies the condition (1.4) and is such that the torsional rigidity of the bar achieves a maximum value, i.e.

$$K \to \max_{\Gamma}. \tag{2.4}$$

This problem is a particular case of the problem (1.1)–(1.5) (coefficients a, b and c are constant), and hence its solution can be found by using the optimality condition (1.14). Solving eqn (1.1), together with the conditions (1.2) and (1.4), we get

$$\phi = \frac{1}{2(ab - c^2)}[\mu - bx^2 - 2cxy - ay^2]$$
$$\Gamma\colon bx^2 + 2cxy + ay^2 = \mu \tag{2.5}$$

In order to determine the constant μ, use is made of the isoperimetric equality (1.4). It is easy to show that

$$\mu = \frac{S}{\pi}\sqrt{(ab - c^2)},$$

and, hence, the optimal solution has the form

$$\phi = \frac{1}{2(ab - c^2)}\left[\frac{S}{\pi}\sqrt{(ab - c^2)} - bx^2 - 2cxy - ay^2\right]$$
$$\Gamma\colon bx^2 + 2cxy + ay^2 = \frac{S\sqrt{(ab - c^2)}}{\pi}. \tag{2.6}$$

The torsional rigidity K_{opt} of the bar with optimal shape of the cross-section (2.6) is given by

$$K_{opt} = \frac{S^2}{2\pi\sqrt{(ab - c^2)}}. \tag{2.7}$$

Let us compare the torsional rigidity of the optimal bar with that of a bar of circular cross-section with the same area S in order to estimate the effectiveness of optimization. For the latter bar, we have

$$K_{\text{cir}} = \frac{S^2}{\pi(a+b)}. \tag{2.8}$$

From (2.7) and (2.8) it follows that

$$\frac{K_{\text{opt}} - K_{\text{cir}}}{K_{\text{cir}}} = 1/\beta - 1; \qquad \beta \equiv \frac{2\sqrt{(ab - c^2)}}{a+b}. \tag{2.9}$$

The constant β can easily be shown to satisfy the inequality $0 \le \beta \le 1$.

It is instructive to specialize the above results to the case of an orthotropic bar, for which $c = 0$, $a = 1/G_2$, $b = 1/G_1$, where G_1 and G_2 denote the shear moduli in the x and y directions, respectively. The optimal shape and the corresponding function ϕ take the form

$$\Gamma\colon G_1x^2 + G_2y^2 = \frac{S\sqrt{(G_1G_2)}}{\pi}$$

$$\phi = \frac{1}{2}\left(\frac{S\sqrt{(G_1G_2)}}{\pi} - G_1x^2 - G_2y^2\right). \tag{2.10}$$

The torsional rigidity K_{opt} of the optimal bar is given by

$$K_{\text{opt}} = \frac{S^2\sqrt{(G_1G_2)}}{2\pi}. \tag{2.11}$$

The relative increase in the torsional rigidity of the optimal orthotropic bar K_{opt} over that of a bar of circular cross-section K_{cir} of the same area S is given by

$$\frac{K_{\text{opt}} - K_{\text{cir}}}{K_{\text{cir}}} = \frac{G_1 + G_2}{2\sqrt{(G_1G_2)}} - 1. \tag{2.12}$$

From (2.12) it is seen, that the optimization becomes more effective both as $G_1/G_2 \to 0$ and $G_1/G_2 \to \infty$. If $G_1 = G_2 = G$, (i.e., the case of a homogeneous isotropic bar under torsion), we have

$$\frac{K_{\text{opt}} - K_{\text{cir}}}{K_{\text{cir}}} = 0.$$

It is known that in this case $K_{\text{opt}} = K_{\text{cir}} = GS^2/2\pi$. In other words, the optimal homogeneous isotropic bar under torsion is circular in shape. This result was proved by Polya[1] through the use of the symmetrization theorem.

It should be noted that for a homogeneous isotropic bar of optimal cross-sectional shape the stress intensity along the boundary of the region D is constant

$$T^2 = T_{xz}^2 + T_{yz}^2 = \theta^2(\phi_x^{\,2} + \phi_y^{\,2}) = \text{Const.},$$

and the optimality condition (1.14) takes the form

$$\left(\frac{\partial\phi}{\partial n}\right)^2 = \lambda^2,$$

where n is the outer normal to the contour Γ which is a special case of the general optimality condition derived by Prager[4].

In an absolutely analogous manner we may consider the dual optimization problem—that of finding the shape of a bar with a minimum cross-sectional area—

$$S = \iint_D \mathrm{d}x\,\mathrm{d}y \to \min_\Gamma \tag{2.13}$$

for a given torsional rigidity

$$K = -\frac{1}{2}\min_\phi J = K'. \tag{2.14}$$

The optimality condition for the problem (2.13)–(2.14) has the same form as before (1.14). Therefore, without going into details, let us only present the final result

$$\Gamma:\ bx^2 + 2cxy + ay^2 = (ab - c^2)^{3/4}\sqrt{(2K'/\pi)}$$

$$S = \sqrt{(2\pi K')}(ab - c^2)^{1/4}.$$

3. OPTIMAL SHAPE OF A BAR WITH A MULTIPLY-CONNECTED CROSS-SECTION

Consider a homogeneous isotropic bar with a doubly-connected cross-section subjected to a twisting moment. Let Γ_0 and Γ denote, respectively, the inner and the outer boundaries of the region D. The stress T_{xz} and T_{yz} may be expressed through a stress function ϕ (see [6])

$$T_{xz} = G\theta\phi_y, \qquad T_{yz} = -G\theta\phi_x, \tag{3.1}$$

which is defined, as usual, by the boundary-value problem

$$\phi_{xx} + \phi_{yy} = -2 \tag{3.2}$$

$$\phi = 0, \qquad (x, y) \in \Gamma \tag{3.3}$$

$$\phi = C, \qquad (x, y) \in \Gamma_0. \tag{3.4}$$

The solution of this boundary-value problem depends on the constant C which is sought by using the condition

$$\int_{\Gamma_0} \frac{\partial\phi}{\partial n}\,\mathrm{d}s = -2\Omega, \tag{3.5}$$

where Ω is the area of the region bounded by the contour Γ_0.

The torsional rigidity K is determined from

$$K = 2\left(\iint_D \phi\,\mathrm{d}x\,\mathrm{d}y + C\Omega\right). \tag{3.6}$$

It should be noted that the expressions for the rigidity K and the function ϕ given above differ from the corresponding expressions of previous section (isotropic material) by the constant multiplier G.

The boundary Γ_0 is assumed to be given, and we are required to find the shape of the boundary Γ for which the functional (3.6) achieves a maximum value subject to the condition (1.4). In order to derive the necessary optimality condition for the present problem one could follow arguments similar to those of Section 1 for a simply-connected region. However, for methodological reasons an alternate procedure is used here. Consider another function ψ, related to ϕ through

$$\psi_x = \phi_y + y; \qquad \psi_y = \phi_x - x. \tag{3.7}$$

The function ψ satisfies Laplace's equation in the region D

$$\psi_{xx} + \psi_{yy} = 0, \tag{3.8}$$

and the following condition on the boundary of the region

$$\frac{\partial\psi}{\partial n} = yn_x - xn_y, \tag{3.9}$$

where $\mathbf{n}_x$ and $\mathbf{n}_y$ denote the direction cosines of a unit normal to $\Gamma_0 + \Gamma$. The torsional rigidity is given by the expression

$$K = I + \iint_D (x\psi_y - y\psi_x)\,dx\,dy, \qquad I = \iint_D (x^2 + y^2)\,dx\,dy. \tag{3.10}$$

For a region D of given shape the function $\psi(x, y)$ may be found by solving the variational problem [6]

$$J_1 = \frac{1}{2}\iint_D \{(\psi_x - y)^2 + (\psi_y + x)^2\}\,dx\,dy \to \min_\psi. \tag{3.11}$$

The extremum of the functional J_1 fulfils the condition $K_* = 2J_1$. Therefore the original optimization problem may be rewritten as

$$K_* = \max_\Gamma \min_\psi 2J_1. \tag{3.12}$$

The minimum of the functional J_1 with respect to ψ is sought among a class of continuously differentiable functions ψ satisfying the boundary condition (3.9). It should be mentioned, however, that this minimum may be sought among a wider class of functions not necessarily satisfying the said boundary condition. Insofar as (3.9) is a "natural" condition for the functional (3.11), the solution of the variational problem found among a class of continuously differentiable functions will "automatically" fulfil this condition. The maximum of the functional J_1 with respect to Γ is sought among closed curves satisfying the isoperimetric condition (1.4). The expression for the first variation of J_1 has the form

$$\delta J_1 = -2\iint_D (\psi_{xx} + \psi_{yy})\delta\psi\,dx\,dy + 2\int_{\Gamma+\Gamma_0}\left(\frac{\partial\psi}{\partial n} - y\mathbf{n}_x + x\mathbf{n}_y\right)\delta\psi\,ds$$
$$+ \int_\Gamma \{(\psi_x - y)^2 + (\psi_y + x)^2\}\delta f\,ds.$$

Putting $\delta J_1 = 0$ and knowing that $\delta\psi$ is arbitrary, we arrive at relations (3.8), (3.9) and the equality

$$\int_\Gamma \{(\psi_x - y)^2 + (\psi_y + x)^2\}\delta f\,ds = 0. \tag{3.13}$$

From (3.13) and the isoperimetric condition

$$\int_\Gamma \delta f\,ds = 0,$$

we get the necessary optimality condition

$$(\psi_x - y)^2 + (\psi_y + x)^2 = \lambda^2, \tag{3.14}$$

or in terms of the original stress function ϕ

$$\phi_x^2 + \phi_y^2 = \lambda^2. \tag{3.15}$$

4. SOLUTION OF THE OPTIMIZATION PROBLEM BY A PERTURBATION TECHNIQUE

In the sequel it is convenient to employ a curvilinear coordinate system s, t related to the contour Γ_0. The coordinate t of a point $Q \in D$ is the distance along the normal QA from the point Q to the contour Γ_0, while s is the distance along the contour measured from a certain fixed point 0 to the point A. In terms of the spatial variables s, t the relations (3.2)–(3.6) take the form

$$
\begin{gathered}
(T\phi_t)_t + (T^{-1}\phi_s)_s = -2T; \qquad T \equiv 1 + \frac{t}{\rho} \\
\phi(s, h) = 0 \\
\phi(s, 0) = C \\
\int_0^L \phi_t(s, 0)\,\mathrm{d}s = -2\Omega \\
\int_0^L \int_0^h T\,\mathrm{d}t\,\mathrm{d}s = S \\
K = 2\left(\int_0^L \int_0^h T\phi\,\mathrm{d}t\,\mathrm{d}s + C\Omega\right),
\end{gathered} \tag{4.1}
$$

where $h = h(s)$ is the equation of the contour Γ, ρ the radius of curvature of the contour Γ_0 and L its length.

4.1. *It will be assumed*, that

$$
\begin{aligned}
&\max_s h(s) = H \ll L \qquad (0 \le s \le L) \\
&\min_s \rho(s) \sim L.
\end{aligned} \tag{4.2}
$$

For the sake of convenience, let us introduce new variables

$$
\begin{aligned}
&s = Ls', \quad t = Ht', \quad h = Hh', \quad \phi = HL\phi' \\
&\Omega = L^2\Omega', \quad S = HLS', \quad \rho = L\rho', \quad K = HL^3K' \\
&C = HLC', \quad \lambda = L\lambda',
\end{aligned} \tag{4.3}
$$

and a small parameter $\epsilon = H/L$.

In terms of the new variables, relations (4.1) and (3.15) take the form (primes have been left out)

$$
\begin{gathered}
(T\phi_t)_t + \epsilon^2(T^{-1}\phi_s)_s = -2\epsilon T; \qquad T = 1 + \frac{\epsilon t}{\rho} \\
\phi_t(s, h) = -\lambda, \quad \phi(s, 0) = -C, \quad \phi(s, h) = 0 \\
\int_0^1 \phi_t(s, 0)\,\mathrm{d}s = -2\Omega, \quad \int_0^1 \left(h + \frac{\epsilon h^2}{2\rho}\right)\mathrm{d}s = S, \\
K = 2\left(C\Omega + \epsilon\int_0^1 \int_0^h T\phi\,\mathrm{d}t\,\mathrm{d}s\right)
\end{gathered} \tag{4.4}
$$

The solution of this problem will be sought in the form of series expansions with respect to the small parameter ϵ

$$
\begin{aligned}
\phi &= \phi^0 + \epsilon\phi^1 + \epsilon^2\phi^2 + \cdots \\
h &= h^0 + \epsilon h^1 + \epsilon^2 h^2 + \cdots \\
K &= K^0 + \epsilon K^1 + \epsilon^2 K^2 + \cdots.
\end{aligned} \tag{4.5}
$$

For finding the zeroth, first and second order approximate solutions it is sufficient to substitute the expansions (4.5) into the relations (4.4) and to equate the coefficients of like powers in ϵ. The resulting boundary-value problems serve to determine the unknown functions. Thus, for determining the unknown functions of zeroth order we have

$$\phi^0_{tt} = 0, \quad \phi^0(s,0) = C^0, \quad \phi^0(s,h^0) = 0 \tag{4.6}$$

$$\phi_t^0(s,h^0) = -\lambda^0, \quad \int_0^1 \phi_t^0(s,0)\,\mathrm{d}s = -2\Omega \tag{4.7}$$

$$\int_0^1 h^0\,\mathrm{d}s = S. \tag{4.8}$$

Similarly the first approximation is the solution of the boundary-value problem

$$\begin{gathered} \phi^1_{tt} = -2 - \frac{\phi_t^0}{\rho}, \quad \phi^1(s,0) = C^1, \quad \phi^1(s,h^0) = 0 \\ \phi_t^1(s,h^0) = -\lambda^1 \\ \int_0^1 \phi_t^1(s,0)\,\mathrm{d}s = 0, \quad \int_0^1 h^1\,\mathrm{d}s = -\frac{1}{2}\int_0^1 \frac{(h^0)^2}{\rho}\,\mathrm{d}s, \end{gathered} \tag{4.9}$$

and the second approximation—that of the problem

$$\begin{gathered} \phi^2_{tt} = -\frac{1}{\rho}[(t\phi_t^1)_t + 2t] - \phi^0_{ss}, \quad \phi^2(s,0) = C^2 \\ \phi^2(s,h^0) = \lambda^1 h^1 + \lambda^0 h^0 \\ \phi_t^2(s,h^0) = \left(2 + \frac{\lambda^0}{\rho}\right)h^1 - \lambda^2 \\ \int_0^1 \phi_t^2(s,0)\,\mathrm{d}s = 0, \quad \int_0^1 h^2\,\mathrm{d}s = -\int_0^1 \frac{h^0 h^1}{\rho}\,\mathrm{d}s. \end{gathered} \tag{4.10}$$

From the zeroth, first and the second order approximate solutions we can define the torsional rigidity of the bar (the functional of the variational problem)

$$K = K^0 + \epsilon K^1 + \epsilon^2 K^2 = 2C^0\Omega + 2\epsilon\left(\int_0^1\int_0^{h^0} \phi^0\,\mathrm{d}t\,\mathrm{d}s + C^1\Omega\right) + 2\epsilon^2\left(\int_0^1\int_0^{h^0} \phi^1\,\mathrm{d}t\,\mathrm{d}s + C^2\Omega\right) + 0(\epsilon^3).$$

Let us proceed with the solution of the above boundary-value problems. We start with the zeroth order terms. From the relations (4.6) and (4.7), we have

$$\phi^0 = C^0\left(1 - \frac{t}{h^0}\right), \quad h^0 = \frac{C^0}{\lambda^0}. \tag{4.11}$$

Substituting (4.11) into the isoperimetric equality (4.8) and performing elementary transformations, we find

$$\lambda^0 = 2\Omega, \quad C^0 = 2S\Omega.$$

Finally, the zeroth order solution takes the form

$$h^0 = S, \quad \phi^0 = 2S\Omega\left(1 - \frac{t}{S}\right), \quad K^0 = 4S\Omega^2. \tag{4.12}$$

Thus, in the zeroth order approximation for hollow bars with fairly shallow (large radius of curvature) inner contours the optimal bar will be of constant thickness.

Similarly, the solution of the boundary-value problem (4.10) furnishes the quantities h^1, ϕ^1, C^1, λ^1 and K^1. For the sake of brevity, we present only the final result

$$h^1 = -\frac{2S^2}{\rho}$$

$$\phi^1 = t^2\left(\frac{\Omega}{\rho} - 1\right) + 2S\Omega\left(\int_0^1 \frac{\mathrm{d}s}{\rho} - \frac{1}{\rho}\right) + S^2 - 2\Omega S^2 \int_0^1 \frac{\mathrm{d}s}{\rho}$$

$$C^1 = S^2\left(1 - 2\Omega \int_0^1 \frac{\mathrm{d}s}{\rho}\right), \quad \lambda^1 = 2S\left(1 - \Omega \int_0^1 \frac{\mathrm{d}s}{\rho}\right)$$

$$K^1 = 4\Omega S^2\left(1 - \Omega \int_0^1 \frac{\mathrm{d}s}{\rho}\right). \tag{4.13}$$

Thus, to within the terms of order ϵ^2, we have

$$h = h^0 + \epsilon h^1 = S\left(1 - \epsilon \frac{S}{2\rho}\right), \tag{4.14}$$

or in terms of the original dimensioned quantities,

$$h = \frac{S}{L}\left(1 - \frac{S}{2\rho L}\right). \tag{4.15}$$

From (4.14) and (4.15) it is evident that the wall thickness of the optimal bar decreases as we move along the inner contour in the direction of increasing curvature.

In order to determine the terms of order ϵ^2, it is necessary to integrate the eqn (4.10) and to choose the arbitrary constants of integration from the boundary and the isoperimetric conditions. Finally, we get

$$h^2 = -\frac{S^3}{2\rho^2}\left(3 + \frac{\rho}{\Omega} - 4\rho^2 \int_0^1 \frac{\mathrm{d}s}{\rho} - \frac{\rho^2}{\Omega} \int_0^1 \frac{\mathrm{d}s}{\rho}\right)$$

$$C^2 = \frac{2S^3}{3}\left(\Omega \int_0^1 \frac{\mathrm{d}s}{\rho^2} - 2\int_0^1 \frac{\mathrm{d}s}{\rho}\right) + 2S^3\Omega\left(\int_0^1 \frac{\mathrm{d}s}{\rho}\right)^2 \tag{4.16}$$

$$K^2 = \frac{2S^3}{3}\left[C\Omega^2\left(\int_0^1 \frac{\mathrm{d}s}{\rho}\right)^2 + 2\Omega^2 \int_0^1 \frac{\mathrm{d}s}{\rho^2} - s\Omega \int_0^1 \frac{\mathrm{d}s}{\rho} + 2\right].$$

Using the expressions for K^0, K^1 and K^2 from (4.12), (4.13) and (4.16) and reverting to the original dimensioned quantities, we get the following expression for the torsional rigidity of the optimal bar

$$K = \frac{2S\Omega^2}{L^2} + \frac{4S^2\Omega}{L^2}\left(1 - \frac{\Omega}{L^2}\int_0^L \frac{\mathrm{d}s}{\rho}\right) + \frac{4S^3}{3L^6}\left[3\Omega^2\left(\int_0^L \frac{\mathrm{d}s}{\rho}\right)^2 + \Omega^2 L \int_0^L \frac{\mathrm{d}s}{\rho^3} - 4\Omega L^2 \int_0^L \frac{\mathrm{d}s}{\rho} + L^4\right]. \tag{4.17}$$

Let us compare the torsional rigidity K of the optimally designed bar with that of a bar of constant thickness ($h(s) = \text{const.}$) having the same inner contour and the cross-sectional area K'. Performing similar calculations as for the optimization problem, we have, to the same degree of accuracy, the following expression for K'

$$K' = \frac{4S\Omega^2}{L^2} + \frac{4S^2\Omega}{L^2}\left(1 - \frac{\Omega}{L^2}\int_0^L \frac{\mathrm{d}s}{\rho}\right) + \frac{4S^3}{3L^6}\left[\frac{15}{4}\Omega^2\left(\int_0^L \frac{\mathrm{d}s}{\rho}\right)^2 - 4\Omega L^2 \int_0^L \frac{\mathrm{d}s}{\rho} + \frac{\Omega^2 L}{4}\int_0^L \frac{\mathrm{d}s}{\rho^2} + L^4\right]. \tag{4.18}$$

From (4.17) and (4.18) we get

$$\Delta K = K - K' = \frac{S^3\Omega^2}{L^6}\left[L\int_0^L \frac{\mathrm{d}s}{\rho^2} - \left(\int_0^L \frac{\mathrm{d}s}{\rho}\right)^2\right]. \tag{4.19}$$

Employing the Schwarz inequality it is easy to show that the expression within the square brackets in (4.19) is always positive, and, therefore, for any inner contour Γ_0 having $\min_s \rho(s) \sim L$ the following inequality is true:

$$\Delta K \geq 0, \tag{4.20}$$

the equality sign holding for the case where Γ_0 is the circumference of a circle (ρ = const. = R, R – radius of the circle). In this case the thin-walled optimal bar is of constant thickness.

4.2. *In Section* 4.1 it was assumed that the minimum radius of curvature is of the order of the length L of the contour Γ_0. In other words, the choice was limited to only weakly curved contours. Consider now the case, where

$$\min_s \rho(s) \sim H$$

everywhere on the contour. Employing again the variables (4.3) the necessary conditions for the solution of the present problem can be obtained by formally replacing ρ in (4.4) by $\epsilon\rho$. Using the perturbation technique we arrive at the following boundary-value problem defining the zeroth order functions

$$\begin{gathered} \left[\left(1+\frac{t}{\rho}\right)\phi_t^0\right]_t = 0, \quad \phi^0(s,0) = C^0, \quad \phi^0(s,h^0) = 0 \\ \phi_t^0(s,h^0) = -\lambda^0 \\ \int_0^1 \phi_t^0(s,0)\,\mathrm{d}s = -2\Omega, \quad \int_0^1 \left[h^0 + \frac{(h^0)^2}{2\rho}\right]\mathrm{d}s = S. \end{gathered} \tag{4.21}$$

The solution of the problem (4.21) gives the following expressions for the stress function $\phi^0(s,t)$ and the wall thickness distribution $h^0(s)$ for the optimal bar

$$\begin{gathered} \phi^0(s,t) = 2\Omega\left[\int_0^1 \frac{\mathrm{d}s}{\rho\ln\left(1+\dfrac{h^0}{\rho}\right)}\right]^{-1} \\ \rho\left(1+\frac{h^0}{\rho}\right)\ln\left(1+\frac{h^0}{\rho}\right) = \text{const.} \end{gathered} \tag{4.22}$$

From the second relation above it is evident that the thickness $h(s)$ decreases as we move along the inner contour Γ_0 in the direction of increasing curvature.

Analogous calculations can be made when certain parts of the inner contour Γ_0 satisfy

$$\rho \sim \epsilon^m H.$$

In this case, it is easy to show that the optimal thickness distribution of the parts with large curvature is related to the radius of curvature of the contour Γ_0 through

$$\rho = h^0 \exp\left(-\frac{\gamma}{h^0}\right),$$

where γ is a constant determined by the isoperimetric equality.

5. CONCLUSIONS

The present work considers optimization problems of mechanical systems described by (elliptical) partial differential equations. The necessary optimality conditions are derived, and analytical solutions of the optimization problems for elastic bars under torsion are found. It is shown that among anisotropic bars the one with an elliptical cross-section has the maximum torsional rigidity, and that the increase in the torsional rigidity over that of a bar of circular cross-section with the same area is bigger, the higher the degree of anisotropy. For thin-walled isotropic bars of a doubly-connected cross-section it is shown that the optimal shape of the outer contour depends significantly on the curvature of the inner contour, the shape of which is assumed to be given. if the inner contour is of small curvature, then in the zeroth order approximation the optimal bar will have a cross-section of constant wall thickness. Moreover, the first and higher order corrections to this solution show that, at the parts of the inner contour that have a larger curvature, there is a certain reduction in the wall thickness of the optimal bar. On the other hand, if the inner contour is of large curvature, the optimal wall thickness distribution, already in the zeroth approximation, is no longer uniform, but diminishes as we move along the inner contour in the direction of increasing curvature.

The possibilities of the analytical procedure are not exhausted by the various types of optimization problems considered in the present work. Likewise, the perturbation technique may be used to solve the problem of optimizing the thickness distribution of a thin-walled bar under torsion with a given outer contour, as well as a bar of multiply-connected cross-section. Various other constraints can be accounted for within the framework of the technique used. Thus, for example, one may optimize the shape of a simply- or multiply-connected cross-section when the shape of a part of the contour is specified and only the rest of the contour can be varied. Such problems naturally arise in connexion with build-up bars.

Finally, it should be noted that the equation of the type (1.1) describes various different physical phenomena. Therefore, by employing available analogies (hydrodynamic, elastic, electrical, etc. [7]), it is possible to interpret the present results accordingly.

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